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LETTER TO THE EDITOR

A note on elliptic coordinates on the Lie algebra $e(3)$

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Abstract
We introduce elliptic coordinates on the dual space to the Lie algebra $e(3)$ and discuss the separability of the Clebsch system in these variables. The proposed Darboux coordinates on $e^*(3)$ coincide with the usual elliptic coordinates on the cotangent bundle of the two-dimensional sphere at the zero value of the corresponding Casimir function.

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The Lie algebra $e(3) = so(3) \oplus \mathbb{R}^3$ of the Lie group of Euclidean motions of $\mathbb{R}^3$ is a semidirect sum of an algebra $so(3)$ and an Abelian ideal $\mathbb{R}^3$. For convenience we shall use the invariant inner product to identify the dual of the Lie algebra, namely $e^*(3)$, with the Lie algebra $e(3)$.

On the dual space $e^*(3)$ with coordinates $J = (J_1, J_2, J_3) \in so(3) \simeq \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ the Lie–Poisson bracket is defined by

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \qquad \{x_i, x_j\} = 0, \tag{1}$$

where ε_{ijk} is the sign of the permutation (ijk) of (123). The Lie–Poisson bracket (1) is degenerated and has two Casimir functions

$$A = |x|^2 \equiv \sum_{k=1}^{3} x_k^2, \qquad B = \langle x, J\rangle \equiv \sum_{k=1}^{3} x_k J_k. \tag{2}$$

Here $\langle x, J\rangle$ stands for the standard Euclidean scalar product in $\mathbb{R}^3$. The generic symplectic leaf,

$$\mathcal{O}_{ab} = \{x, J : A = a^2, B = b\}, \tag{3}$$

is a four-dimensional symplectic manifolds, which is topologically equivalent to the cotangent bundle $T^*\mathbb{S}^2$ of the two-dimensional sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3, |x| = a\}$ [1].

If $b = 0$, then there exists a symplectic transformation

$$\rho : (p, x) \to J = p \wedge x, \qquad J_i = \sum_{j,k=1}^{n=3} \varepsilon_{ijk} p_j x_k, \tag{4}$$

which identify $T^*\mathbb{S}^2 \subset T^*\mathbb{R}^3$ and $\mathcal{O}_{a0}$. Here $p \wedge x$ means the standard Euclidean cross product in $\mathbb{R}^3$ and vector p is canonically conjugated to x momenta in $T^*\mathbb{R}^3$, $\{p_i, x_j\} = \delta_{ij}$, such that $\langle p, x\rangle = 0$.

If $b \neq 0$, the symplectic structure of manifold $\mathcal{O}_{ab}$ differs from the standard symplectic structure of $T^*\mathbb{S}^2$ by a magnetic term proportional to b [1].

The aim of this letter is to describe this magnetic term with the help of elliptic coordinates on the sphere $\mathbb{S}^2$ lifted to the Darboux variables on the manifold $e^*(3)$ by $b \neq 0$.

The elliptic coordinates u_1, u_2 on $\mathbb{S}^2$ with parameters $\alpha_1 < \alpha_2 < \alpha_3$ are defined as roots of the equation

$$e(\lambda) = \sum_{j=1}^{3} \frac{x_j^2}{\lambda - \alpha_j} = \frac{a^2(\lambda - u_1)(\lambda - u_2)}{\varphi(\lambda)} = 0, \tag{5}$$

where $\varphi(\lambda) = \prod_{j=1}^{3}(\lambda - \alpha_j)$ and $|x| = a$, see [2]. Like the elliptic coordinates in $\mathbb{R}^3$, the elliptic coordinates on $\mathbb{S}^2$ are also orthogonal and only locally defined. They take values in the intervals

$$\alpha_1 < u_1 < \alpha_2 < u_2 < \alpha_3. \tag{6}$$

By using the Lie–Poisson bracket (1) we can prove that elliptic coordinates $u_{1,2}$ and variables

$$\pi_{1,2}^0 = h(u_{1,2}), \qquad h(\lambda) = \frac{1}{2a^2} \sum_{j=1}^{3} \frac{x_j (x \wedge J)_j}{\lambda - \alpha_j} \tag{7}$$

satisfy the following relations:

$$\{u_1, u_2\} = 0, \qquad \left\{\pi_i^0, u_j\right\} = \delta_{ij}, \qquad \left\{\pi_1^0, \pi_2^0\right\} = \frac{\mathrm{i}b}{4a} \frac{u_2 - u_1}{\sqrt{\varphi(u_1)\varphi(u_2)}}. \tag{8}$$

These relations may be easily obtained by means of the Poisson brackets between the generating functions $e(\lambda)$ and $h(\mu)$:

$$\{e(\lambda), e(\mu)\} = 0, \qquad \{e(\lambda), h(\mu)\} = -a^{-2} e(\lambda) e(\mu) - \frac{e(\lambda) - e(\mu)}{\lambda - \mu},$$

$$\{h(\lambda), h(\mu)\} = \frac{b}{4a^2} \frac{x_1 x_2 x_3 (\lambda - \mu)(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}{\varphi(\lambda)\varphi(\mu)}.$$

If $b = 0$ relations (8) yield the well-known fact that variables $u_{1,2}$ and $\pi_{1,2}^0$ are the Darboux coordinates on the manifold $T^*\mathbb{S} \simeq \mathcal{O}_{a0}$.

The coordinates u_i and the parameters α_j can be subjected to a simultaneous linear transformation $u_i \to \beta u_i + \gamma$ and $\alpha_j \to \beta \alpha_j + \gamma$, so it is always possible to choose

$$\alpha_1 = 0, \qquad \alpha_2 = 1, \qquad \alpha_3 = \boldsymbol{k}^2 > 1.$$

Using relations (8) and this choice of parameters α_j we can prove the following:

Proposition 1. *If $b \neq 0$, elliptic coordinates $u_{1,2}$ (5) and the corresponding momenta*

$$\pi_{1,2} = \pi_{1,2}^0 - b f_{1,2}, \qquad f_{1,2} = \frac{u_{1,2}}{2a\sqrt{\varphi(u_{1,2})}} F\left(\frac{\sqrt{\boldsymbol{k}^2 - u_{2,1}}}{\boldsymbol{k}}, \frac{\boldsymbol{k}}{\boldsymbol{k}^2 - 1}\right) \tag{9}$$

form a complete set of the Darboux variables on the manifold $e^*(3)$

$$\{u_1, u_2\} = \{\pi_1, \pi_2\} = 0, \qquad \{\pi_i, u_j\} = \delta_{ij},$$

which are real variables in their domain of definition (6). Here $F(z, k)$ is incomplete elliptic integral of the first kind, which is identical to the inverse function of the elliptic Jacobi function $sn(z, k)$ [3].

Proof. The functions $f_{1,2}$ depend on the coordinates $u_{1,2}$ and, therefore, we have to verify one relation only

$$\{\pi_1, \pi_2\} = \{\pi_1^0, \pi_2^0\} - b\{\pi_1^0, f_2\} + b\{\pi_2^0, f_1\} = \{\pi_1^0, \pi_2^0\} - b\left(\frac{\partial f_2}{\partial u_1} - \frac{\partial f_1}{\partial u_2}\right) = 0.$$

This relation follows from (8) and properties of the incomplete elliptic integral of the first kind $F(z, k)$. □

So, equations (5), (7) and (9) define elliptic variables u and π on $e^*(3)$ as functions on initial variables x and J. The inverse transformation $(u, \pi) \to (x, J)$ reads as

$$x_j = a\sqrt{\frac{(\alpha_j - u_1)(\alpha_j - u_2)}{(\alpha_j - \alpha_m)(\alpha_j - \alpha_n)}}, \qquad J_j = a^{-2}(bx_j + (z \wedge x)_j), \tag{10}$$

where $m \neq j \neq n$ and entries of the vector z are given by

$$z_j = \frac{2x_j}{u_1 - u_2}\left(\frac{\varphi(u_1)(\pi_1 + bf_1)}{\alpha_j - u_1} - \frac{\varphi(u_2)(\pi_2 + bf_2)}{\alpha_j - u_2}\right).$$

Like the elliptic coordinates in $\mathbb{R}^n$ and on $\mathbb{S}^n$ [2], the elliptic variables on $e^*(3)$ may be successfully exploited in the theory of integrable systems. For instance, substituting expressions (10) into the quadratic Hamiltonian

$$H = \tfrac{1}{2}\left(J_1^2 + J_2^2 + J_3^2\right) + \tfrac{1}{2}\left(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2\right),$$

associated with the integrable Clebsch system on $e^*(3)$, one gets this Hamiltonian in terms of the elliptic variables

$$H = \frac{2}{u_1 - u_2}(\varphi(u_1)(\pi_1 + bf_1)^2 - \varphi(u_2)(\pi_2 + bf_2)^2) + \frac{b^2}{2a^2} - \frac{a^2\left(u_1 + u_2 - \sum_{j=1}^3 \alpha_j\right)}{2}.$$

Here $f_{1,2}$ are functions on u_1 and u_2 and, therefore, the Hamiltonian H belongs to the Stäckel family of integrals of motion at $b = 0$ only, i.e. for the Neumann system on the sphere [2]. Moreover, $f_{1,2}$ are elliptic functions and, therefore, this Hamiltonian H cannot be rewritten as rational quasi-Stäckel Hamiltonian introduced in [4].

Thus we have to fully appreciate that elliptic variables $u_{1,2}$ and $\pi_{1,2}$ cannot be the separation variables for the Hamilton–Jacobi equation associated with the Clebsch system at $b \neq 0$. The similar result has been obtained in [5, 6] by using velocities

$$\dot{u}_{1,2} = \frac{\mp 4\varphi(u_{1,2})(\pi_{1,2} + bf_{1,2})}{u_1 - u_2}$$

instead of the momenta $\pi_{1,2}$. This result completely refutes the conclusion of the paper [7] about separability of the Clebsch system in the 'Kowalevski variables', which in fact coincide with the elliptic coordinates (5).

On the other hand, we can substitute Darboux variables $u_{1,2}$ and $\pi_{1,2}$ into the usual Stäckel integrals of motion and get a whole family of integrable systems on the manifold $e^*(3)$, which are separable in these variables. The main problem in this widely known Jacobi method is how to single out integrable systems interesting in physics from this huge family.

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